MA 1118 - Multivariable Calculus
Exam III - Quarter I - AY 02-03

Instructions: Work all problems. Read the problems carefully. Show appropriate work, as partial credit will be given. One page 8-1/2x11, one side hand written notes allowed. No graphing or scientific calculators permitted.

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1. (10 points) Show that $\displaystyle \lim_{(x,y,)\to(0,0)} \frac{x^2+3y}{y}$ does not exist.

solution:

Since we're already told that the limit does not exist, they best way to approach this problem is to show that the function approaches different limits as we approach the origin on different paths.

We start by determining whether a limit exists as we approach along straight lines, i.e. curves of the form

$$y = k\ , \quad k = \text{constant}$$

On these,

$$\lim_{(x,y,)\to(0,0)} \frac{x^2+3y}{y} = \lim_{(x,y,)\to(0,0)} \frac{x^2+3kx}{kx} = \lim_{(x,y,)\to(0,0)} \frac{x+3k}{k} = \frac{3k}{k} = 3$$

and therefore, along these curves, a single limit does exist. So we must try some other paths. For example, if we approach the origin along parabolas, i.e. curves of the form

$$y = kx^2\ , \quad k = \text{constant}$$

then, on these,

$$\lim_{(x,y,)\to(0,0)} \frac{x^2+3y}{y} = \lim_{(x,y,)\to(0,0)} \frac{x^2+3kx^2}{kx^2} = \lim_{(x,y,)\to(0,0)} \frac{1+3k}{k}$$

which clearly depends on k, and therefore on the direction of approach. Therefore

$$\lim_{(x,y,)\to(0,0)} \frac{x^2+3y}{y} \qquad \text{does not exist.}$$

2. (25 points) Given

$$f(x,y,z) = \sqrt{x^2+y^2} + \ln\left(e^{yz}+1\right)$$

find $\quad \dfrac{\partial f}{\partial x} \quad , \quad \dfrac{\partial f}{\partial z} \quad$ and $\quad \dfrac{\partial^2 f}{\partial y \partial z}$

solution:

To find $\frac{\partial f}{\partial x}$, we treat all other variables (in this case y) as constants, and then just take an ordinary derivative, i.e.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\left(x^2+y^2\right)^{\frac{1}{2}} + \ln\left(e^{yz}+1\right)\right] \\ &= \frac{1}{2}\ \left(x^2+y^2\right)^{-\frac{1}{2}}(2x) + 0 = \frac{x}{\sqrt{x^2+y^2}}\end{aligned}$$

where we differentiated the first term using just the chain rule, and the second term using the constant rule (since x does not appear anywhere in that term).

Proceeding similarly then

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\left(x^2+y^2\right)^{\frac{1}{2}} + \ln\left(e^{yz}+1\right)\right] = 0 + \frac{1}{e^{yz}+1}\ \left(ye^{yz}\right) = \frac{ye^{yz}}{e^{yz}+1}$$

using the chain rule on the second term, and noting that z does not appear anywhere in the first term.

Finally,

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial z} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial z}\right] = \frac{\partial}{\partial y}\left[\frac{ye^{yz}}{e^{yz}+1}\right] \\ &= \frac{\left[(1)e^{yz} + y\left(ze^{yz}\right)\right]\left[e^{yz}+1\right] - \left[ye^{yz}\right]\left[ze^{yz}\right]}{\left(e^{yz}+1\right)^2} = \frac{e^{yz}\left(e^{yz}+1+yz\right)}{\left(e^{yz}+1\right)^2}\end{aligned}$$

3. (20 points) a. Find the linearization of:

$$f(x,y) = e^{-3x}\cos(2y) + \sin(2y)$$

about $x_0 = 0,\ y_0 = 0$

solution:

By definition, the linearization of $f(x,y)$ about the point (x_0, y_0) is:

$$f(x,y) \doteq f(x_0,y_0) + f_x(x_0,y_0)\ (x - x_0) + f_y(x_0,y_0)\ (y - y_0)$$

But here $\quad f(x_0,y_0) = f(0,0) = e^{-3(0)}\cos(2(0)) + \sin(2(0)) = 1$

Furthermore

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{-3x}\cos(2y) + \sin(2y)\right] = -3e^{-3x}\cos(2y)$$

$$\Longrightarrow \qquad f_x(0,0) = (-3)e^{-3(0)}\cos(2(0)) = -3$$

and finally

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{-3x}\cos(2y) + \sin(2y)\right] = -2e^{-3x}\sin(2y) + 2\cos(2y)$$

$$\Longrightarrow \qquad f_y(0,0) = -2e^{-3(0)}\sin(2(0)) + 2\cos(2(0)) = 2$$

and so the linearization of $f(x,y)$ about $(0,0)$ is:

$$f(x,y) \doteq 1 + (-3)(x-0) + (2)(y-0) = -3x + 2y + 1$$

b. Use your answer from part a. to estimate the value of $f(0.05, 0.04)$.

solution:

Based on the result from part a.,

$$f(0.05, 0.04) \doteq 1 + (-3)(0.05) + (2)(0.04) = -.15 + .08 + 1 = .93$$

(Note that $f(0.05, 0.04) = .93786987...$, and therefore the approximation is accurate to within about one percent.)

4. (20 points) Consider the function of two variables given by

$$f(x, y) = 4x^2 + y^2 + 8 \ .$$

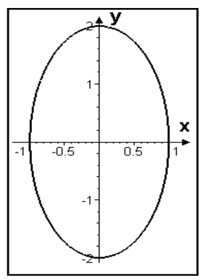
a. Sketch the level curve corresponding to $f(x, y) \ = \ 12$.

solution:

The equation corresponding to the level curve $f(x, y) = 12$ is give by the equation

$$f(x, y) = 4x^2 + y^2 + 8 = 12 \qquad \Longrightarrow \qquad 4x^2 + y^2 = 4$$

This is clearly an ellipse, with semi-major axis equal two up the y-axis, and semi-minor axis of one along the x-axis. Therefore the sketch of this level curve is:



b. Find the (directional) derivative of this function at the point $\mathbf{P}_0 = (1, 2)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$.

solution:

By definition, the directional derivative of the function $f(x, y)$ in the direction of a given vector $\mathbf{v}$ at (x_0, y_0) is

$$\frac{df}{ds} = \nabla f(x_0, y_0) \cdot \frac{\mathbf{v}}{\| \mathbf{v} \|} \quad \text{where} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

But for $f(x, y) = 4x^2 + y^2 + 8$, then

$$\nabla f = 8x\mathbf{i} + 2y\mathbf{j} \qquad \Longrightarrow \qquad \nabla f(1, 2) = 8\mathbf{i} + 4\mathbf{j}$$

solution:

Therefore, at $(1, 2)$,

$$\frac{df}{ds} = (8\mathbf{i} + 4\mathbf{j}) \cdot \frac{\mathbf{i} + 2\mathbf{j}}{\| \, \mathbf{i} + 2\mathbf{j} \, \|} = \frac{16}{\sqrt{1^2 + 2^2}} = \frac{16}{\sqrt{5}} = \frac{16\sqrt{5}}{5}$$

c. Find the direction of the normal to the surface

$$z = f(x, y) = 4x^2 + y^2 + 8$$

at the point $(2, 1, 25)$.

solution:

The surface $z = 4x^2 + y^2 + 8$ is a level surface of

$$F(x, y, z) = z - 4x^2 - y^2$$

corresponding to $F(x, y, z) = 8$. We know the gradient vector

$$\nabla F = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = -8x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$$

is always normal to the level surface of $F(x, y, z)$. Therefore, since

$$\nabla F(2, 1, 215) = -16\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

is normal to the level surface at the given point. However, by definition, the direction of any vector is a *unit vector* in the same direction, and so the direction of the normal here is

$$\mathbf{n} = \frac{\nabla F(2, 1, 25)}{\| \, \nabla F(2, 1, 25) \, \|} = \frac{-16\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\| \, -16\mathbf{i} - 2\mathbf{j} + \mathbf{k} \, \|} = \frac{-16\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\sqrt{(-16)^2 + (-2)^1 + 1^2}}$$

$$= \frac{-16\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\sqrt{261}}$$

5. (25 points) Find and correctly identify all the local maxima, minima and saddle points of

$$f(x,y) = x^4 - 16xy + 8y^2 + 7$$

solution:

The first step is to identify all critical points. Since $f(x,y)$ is obviously differential everywhere (it's a polynomial), then these are precisely the solutions of:

$$\frac{\partial f}{\partial x} = 4x^3 - 16y = 0$$

$$\frac{\partial f}{\partial y} = -16x + 16y = 0 \qquad \Longrightarrow \qquad x = y$$

Substituting the result of the second equation into the first yields

$$\frac{\partial f}{\partial x} = 4x^3 - 16x = 4x(x^2 - 4) = 0 \qquad \Longrightarrow \qquad x = 0,\ \pm 2$$

Therefore, we have three critical points, $(0,0)$, $(2,2)$ and $(-2,-2)$.

Now set up the "standard" table

x	y	$f_{xx} = 12x^2$	$f_{yy} = 16$	$f_{xy} = -16$	$f_{xx}f_{yy} - f_{xy}^2$	Type
0	0	0	16	-16	$(0)(16) - (-16)^2$ $= -256 < 0$	saddle
2	2	48	16	-16	$(48)(16) - (-16)^2$ $= 512 > 0$	min
-2	-2	48	16	-16	$(48)(16) - (-16)^2$ $= 512 > 0$	min

where both $(2,2)$ and $(-2,-2)$ are local minima because either $f_{xx} > 0$ there. (Or, alternatively, because $f_{yy} > 0$ there.)